within our universities and colleges, and this fact can be traced to a neglect upon the part of academic men and women of the form in which they arrange their material. *

It is widely recognized that such a neglect of form as I have indicated appears in much of our university lecturing. The assumption of the ordinary university lecturer is that if he presents a certain body of material so organized that it seems to him to be fairly coherent and logical in its character, it is a matter of small moment whether it appeals to the students because of its literary form or whether it is easily intelligible to them because of its careful adjustment to their present stage of developement. * *

The elective system has in part overcome this attitude and there is a very much more general conviction on the part of college instructors now than there was a generation ago that the demands of the student that the material shall be presented in clear and coherent form should be met. An instructor who is in competition with the other members of his own department for students in his course is likely to recognize the importance of preparing his material as clearly as possible for presentation to his class. * *

It is obvious that everyone who is successful in the art of teaching must have complied with the demands indicated in the foregoing discussion; that is, he must have organized his material in such a way that it has significance not only for his own mind but also for the minds of others. The teacher who does not sympathize with his pupils fails commonly because he does not recognize the type of fact which we have just been discussing.

It should not be inferred from the opening sentence above quoted that all of the five hundred Doctors of the University of Chicago look with distrust upon the proposition that a university man who expects to teach in a college or university should know something of the principles, history and practice of education in general and much of these things as they pertain to his own department in particular, in order that he may, at least, avoid the historic blunders of his predecessors. The following quotation from the symposium bears upon this point:

As matters now stand, whatever the *ideal* held by the university world in regard to the high calling of the doctor in the field of pure research, the *fact* is that the great majority of the doctors now turned out in this country are nominated and pushed by their respective departments, for teaching positions, which they must fill successfully or be counted as failures by all who measure the ratio of results accomplished to tasks undertaken.

Success in teaching is well-nigh *indispensable* (save perhaps in a great university, where failure is less conspicuous because of the large numbers and multifarious interests). If the doctor is to gain encouragement and opportunity to go on with his research, he must first establish a reputation for himself in the community where he goes, for soundness of judgment, clearness of presentation, and power to inspire and lead students; then, having made sure of his ground, he can have pretty much his own way in planning and carrying out his work in the interest of his research. But, on the contrary, if in his blind devotion to his ideal he fails at the outset to teach successfully, to inspire and to lead students, he thereby cuts himself off from his very best resources for ultimately accomplishing his highest aims.

These being the facts as realized in everyday experiences, the conclusion is forced upon us that the universities must prepare for teaching those doctors whom it is proposed to recommend as teachers, and preparations must include breadth of culture, eliminations of angularities, development of pedagogical sense, and some acquaintance with the great educational movements of the past and present. If it be urged that these, some or all of them, are either matters of personal quality or are foreign to the great purpose of the graduate school as director of explorations in unknown fields, then the other conclusion is forced upon us that only those doctors should be recommended as teachers who possess by

nature the prerequisities or have developed them *outside* or *in spite of* the graduate school, and that the present indiscriminate practice of recommending any doctor as a teacher, however narrow or however lacking in the elements of pedagogic sense and power to teach, be superseded by a careful discrimination and selection of those who are prepared to teach and a refusal to nominate or recommend those who are not.

Such a discrimination will lead, as it should, to a more careful and personal consideration of every individual's candidacy for the doctorate, and will compel a readjustment of curriculum and a recognition as a distinct field of that broad, thorough, and scientific training, commensurate with any standard now set up for the doctorate, which is demanded by the man or woman as a preparation for the highest attainment in teaching, leaving undisturbed the narrowest and possibly the deepest channels of pure research to be followed by those who either have not the time or have not the inclination, or perchance have not the personal qualities needed in preparation for teaching.

This leads again and finally to the conclusion either that the number of those who are encouraged to go on to the doctorate should be greatly diminished, or else that the basis of the doctorate should be greatly broadened so as to provide the highest standards and the strongest preparation for the noble art of teaching, as well as to produce fine investigators.

Whatever may have been the attitude of the universities up to this time in regard to training men as teachers for college positions, it is clear that from this time forward the demand is likely to increase for men thus trained, not only because the colleges themselves are awakening to the necessity of better teaching, but also because the irresistible wave of progress in this respect in the secondary schools is bound to reach up into the colleges and shame them into action. Fortunately, however, the agitation is already begun to some extent with respect to the colleges, under the challenge of such men as Abraham Flexner and with the support of many who are loyal to the cause of better teaching.

But while the question of better training for teachers of mathematics is sure to get consideration along with the like questions for all other departments in the general upward movement, it is the special responsibility of the mathematicians themselves to consider the subject-matter of the collegiate curriculum, with respect to its better arrangement, better form of presentation, closer contact with concrete applications, closer correlation with related departments, such as Physics, Astronomy, Chemistry, Geology, and better adjustment to both the earlier and later work in the department. In this connection also belongs the consideration of all these questions concerning mathematics as related to students in the literary courses, in the general science courses, in the specialized science courses, in the small college and in the great university.

It is well known that changes have been going on in respect to the status of various branches of mathematics in the college curriculum; for instance, as to the content of College Algebra, the scope of Trigonometry and of Analytic Geometry, the proper adjustment of theory and applications in the Calculus, and the whole question as to whether all of these branches should not be considered as one subject—Mathematics—rather than as sepa-

rate topics carefully partitioned off from each other both theoretically and pedagogically.

The editors of the MONTHLY have decided to open its columns to contributions on the teaching of collegiate mathematics, in the hope that some impetus may be given to the better training of teachers, the better arrangement and coordination of material and the better form and methods of presentation. Already some important papers have been promised and others will be announced in the near future.

Moreover, no less than six sub-committees, under Klein's International Commission on the Teaching of Mathematics, are now working in this country on topics connected with mathematics of a collegiate grade, and it is supposed that the substance of their reports will be made known in America and will become the basis of discussion at the various meetings of mathematical societies during the Autumn and Winter. From these sources also much should be expected both in pointing out existing conditions and in arousing interest in questions of possible improvements.

ON A FEW POINTS IN THE HISTORY OF ELEMENTARY MATHEMATICS.

By G. A. MILLER, University of Illinois.

The main object of the present note is to call attention to the recent changes of view in reference to several important questions in the history of elementary mathematics. Among the general works on the history of mathematics there is probably none which is more commonly trusted than Cantor's *Vorlesungen über Geschichte der Mathematik.* The first volume of this work is now in its third edition, the second and third volumes are in second editions, while the first edition of the fourth volume appeared as recently as 1908. As the first edition of the first volume appeared in 1880, the rapid succession of new editions of such an extensive work is an index of the rapid recent progress in the development of the history of mathematics. Discoveries have followed each other in such close succession as to call for rapid changes of view even in regard to some of the most fundamental matters.

In support of this statement we shall give a few changes of view as exhibited in the second and third editions of volume I of Cantor's monumental work. On page 576 of the second edition the following words may be found, "According to our opinion the discovery of zero is due to the Hindus." The corresponding statement in the third edition, page 616, reads as follows: "According to our opinion the discovery of zero is due to the Baby-

lonians, the deepening of the concept is due to the Hindus." It need scarcely be added that the former of these two views is expressed in nearly all works of reference relating to this subject, but it is likely that this will gradually be changed on account of the great influence of Cantor in historical matters and the strong reasons advanced by him in support of his change of view.

The discovery of zero, as used above, implies its use in positional arithmetic. It is certain that the Greeks employed zero in the second century B. C. to denote the absence of degrees, minutes, or seconds in their sexagesimal notation.* The earliest known use of this symbol in Babylonian inscriptions belongs to the third century B. C., but it is supposed that it was in use at a much earlier date. At the international mathematical congress held in Paris in 1900 Cantor suggested that zero was probably in use among the Babylonians as early as 1700 B. C. Even if such an early date cannot be established it appears likely that scholars will hereafter attribute the discovery of positional arithmetic to the Babylonians instead of to the Hindus.

Another important change of view exhibited in the two editions mentioned above relates to the sexagesimal system of notation, which is still used by us in measuring time and angles. In the second edition, page 92, Cantor says that the ancient Babylonian astronomers probably thought the year was composed of 360 days and hence they divided the circle into 360 parts, each part corresponding to a day of the year. An apparent corroboration of this view was furnished by an ancient Chinese custom to divide the circle into $365\frac{1}{4}$ degrees. After the circle was divided into 360°, it is easy to see that the base 60, for the sexagesimal system, might have been suggested by the fact that the side of a regular inscribed hexagon is equal to the radius of the circle.

In the third edition Cantor abandons this view and suggests that the base 60 may have resulted from the union of two nations, meeting in Babylon, one having a system of numeration with 10 as a base while the other employed 60 as a base. Among the reasons advanced for abandoning his former position are the following: The counting and writing of small numbers must have been known long before such comparatively large numbers as 360 were in use, and it must have been recognized by the early Babylonian astronomers that the year involves more than 360 days. Hence it does not appear likely that there is any connection between the number of degrees in a circle and the supposed number of days in the year.

These changes of view should influence the high school teacher to supplement the historical notes even in many of the best and most recent text-books. Whenever the teacher expresses an opinion in reference to a question which has not been fully settled this opinion should be in accord with the most advanced scholarship. The changes of view of such an eminent scholar as Moritz Cantor should also impress the teacher with the fact

*Encyclopedie des Sciences Mathematiques, Vol. 1, page 17.

that the most modern works should be consulted when one is dealing with historical data. What the foremost scholar along a certain line endorsed ten years ago may be antiquated rubbish to-day.

A less positive but scarcely less important change of view, exhibited in the two editions under consideration, relates to the development of Egyptian mathematics at the time of Ahmes, about 1700 B. C. As the extraction of the square root does not occur in the work of Ahmes, it was natural to infer that the Egyptians were unacquainted with this operation at this early date. In the third edition we are told, on the contrary, that the ancient Egyptians knew how to extract the square root and even how to solve such systems of simultaneous equations as

$$x^2+y^2=100$$
$$x : y=1 : \tfrac{3}{4}.$$

These recent discoveries prove that the ancient Egyptians knew much more about equations than what was previously supposed.

Recent discoveries in Babylon have also led to changes of view. Many of the histories of mathematics state that the Babylonians did not use any number as large as a million and some historians have devoted considerable attention to this supposed fact. The recent discoveries by Hilprecht of the University of Pennsylvania have revealed that the Babylonians made use of as large numbers as

$$195\,,955\,,500\,,000\,,000$$

and that numbers exceeding a million were frequently employed.* Even in the latest edition of Cantor's *Vorlesungen* the obsolete view is expressed.

In closing this note it may be in place to quote the remarks of J. J. Milne before a meeting of British teachers of mathematics, as follows: "Speaking as a schoolmaster to schoolmasters I think we ought to bring the history of mathematics more than we do before the notice of our pupils. Mathematics is a living, growing science, with a definite history, and there is not a branch of it which boys take up in school, whether arithmetic or algebra, or geometry, or trigonometry, or any other of the many divisions, but has its own history, and I have always found that boys are interested in learning what properties were known to the ancients, and what have been discovered in modern times, and I often think that the writers of text-books would do well to devote a little more space than they do to what I may call the note of human interest." For instance, in teaching complex fractions it is of interest to observe that the Hindus of the ninth century taught the rule of inverted divisor and that this rule was rediscovered in Europe in the sixteenth century.

*Smith, *Bulletin of the American Mathematical Society*, vol. 13, page 394.

ON CERTAIN FUNCTIONAL EQUATIONS.*

By R. D. CARMICHAEL, Princeton University.

1. Let $h(x)$ be a uniform analytic function regular in the neighborhood of the origin and defined by the equation

$$h(x+y).h(x-y)=h(x)^2+h(y)^2-c^2,\ c\neq 0. \tag{1}$$

If we put $x=y=0$, we have

$$h(0)^2=2h(0)^2-c^2,\text{ or }\pm h(0)=c. \tag{2}$$

If now we put $\pm h(x)=cf(x)$, the preceding relations become

$$f(x+y).f(x-y)=f(x)^2+f(y)^2-1, \tag{3}$$
$$f(0)=1. \tag{4}$$

Hence, putting $x=y$,

$$f(2x)=2f(x)^2-1. \tag{5}$$

Then
$$f'(2x)=2f(x)f'(x). \tag{6}$$

Differentiating (3) with respect to x and then with respect to y,

$$f'(x+y).f(x-y)+f(x+y).f'(x-y)=2f(x)f'(x),=f'(2x),\text{ by (6)}; \tag{7}$$
$$f'(x+y).f(x-y)-f(x+y).f'(x-y)=2f(y)f'(y),=f'(2y),\text{ by (6)}. \tag{8}$$

If we put $x+y=A$, $x-y=B$, whence $2x=A+B$, $2y=A-B$, and substitute in equations (7) and (8), and finally in the result replace A and B by x and y, respectively, we have the equations

$$f'(x+y)=f'(x)f(y)+f(x)f'(y); \tag{9}$$
$$f'(x-y)=f'(x)f(y)-f(x)f'(y). \tag{10}$$

Adding (9) and (10) and integrating the result with respect to x we obtain

$$f(x+y)+f(x-y)=2f(x)f(y); \tag{11}$$

for from equation (4) it follows that the constant of integration is zero.

We will now show how to revert from equation (11) to equations (1) and (3), when $f(0)\neq 0$. Differentiate (11) with respect to x and then with

*Presented to the American Mathematical Society, April 24, 1909.

respect to y. From the addition and the subtraction of the two resulting equations follow (9) and (10), respectively. In (9) put $x=y$; the result is equation (6). Then, if in (9) and (10) we put $x+y$ for x and $x-y$ for y and employ (6), we obtain (7) and (8). If we integrate (7) and (8) with respect to x and y, respectively, combine the results and choose $-a$ as a constant of integration, we have $f(x+y)f(x-y)=f(x)^2+f(y)^2-a$, where $a\neq 0$ since $f(0)\neq 0$.

From the foregoing considerations it follows that, except possibly for the appearance of a constant factor, equations (1), (3), (11) have the same solution (or solutions) when each alone is taken as a single functional equation, the condition $f(0)\neq 0$ being satisfied. Moreover, either the system (9) and (10) or the system (7) and (8) has the same solution if $f(0)\neq 0$. Then if general f is found subject to the above equations and multiplied by a proper constant, the result is the general solution of (1), (3), (11), (7) and (8), or (9) and (10).

We proceed now to find general f in equation (3). All the equations (4) to (11) depend on (3) and the fact that $f(x)$ is differentiable; and we may therefore make use of any of them.

We first show that $f(x)$ is an even function. Putting $y=-x$ in (3) we have

$$f(2x)=f(x)^2+f(-x)^2-1.$$

Comparing with (5),

$$f(x)^2=f(-x)^2;\ \text{or}\ f(x)=f(-x),\ \text{since}\ f(0)\neq 0.$$

Since $f(0)=1$ and $f(x)$ is an even function regular in the neighborhood of the origin, it may be expanded in series in the form

$$f(x)=1+\frac{ax^2}{2!}+\frac{bx^4}{4!}+\frac{cx^6}{6!}+\frac{dx^8}{8!}+\ldots.$$

Substituting in (5) to determine coefficients, we have $b=a^2$, $c=a^3$, $d=a^4$, ... Hence, writing k^2 for a, we have

$$f(x)=\tfrac{1}{2}(e^{kx}+e^{-kx}), \tag{12}$$

as the general solution of (3) subject to the imposed conditions. Then

$$h(x)=\pm\frac{c}{2}(e^{kx}+e^{-kx}) \tag{13}$$

is the solution of (1) under like conditions. This completes the discussion for the case $c\neq 0$.

If $c=0$, equations (1) and (2) become

$$h(x+y).h(x-y)=h(x)^2+h(y)^2, \qquad (14)$$
$$h(0)=0.$$

Differentiating with respect to x and then with respect to y, we have

$$h'(x+y).h(x-y)+h(x+y).h'(x-y)=2h(x)h'(x),$$
$$h'(x+y).h(x-y)-h(x+y).h'(x-y)=2h(y)h'(y).$$

Adding, putting $y=x$, and remembering that $h(0)=0$, we have

$$h(x).h'(x)=0.$$

Hence $h(x)=0$ or $h'(x)=0$. In either case $h(x)$ is a constant. Since $h(0)=0$, that constant is zero. Hence $h(x)=0$, and the case is trivial. Therefore the function $\frac{1}{2}(e^{kx}+e^{-kx})$, multiplied by a proper constant, constitutes the significant solution of our equations. The value of $f(0)$ determines this multiplicative constant.

2. Consider the equation

$$g(x+y).g(x-y)=g(x)^2-g(y)^2. \qquad (15)$$

If we put $x=y=0$, we have $g(0)=0$. Differentiating with respect to x and then with respect to y,

$$g'(x+y).g(x-y)+g(x+y).g'(x-y)=2g(x)g'(x), \qquad (16)$$
$$g'(x+y).g(x-y)-g(x+y).g'(x-y)=-2g(y)g'(y). \qquad (17)$$

Making $x=y$ and remembering that $g(0)=0$, we have

$$g(2x)g'(0)=2g(x)g'(x).$$

In view of this relation the product of (16) and (17) gives

$$g'(x+y)^2.g(x-y)^2-g(x+y)^2g'(x-y)^2=-g'(0)^2g(2x)g(2y);$$
or, $$g(x)^2g'(y)^2-g'(x)^2g(y)^2=g'(0)^2.g(x+y).g(x-y),$$

on replacing $x+y$ by x, $x-y$ by y, $2x$ by $x+y$, $2y$ by $x-y$. Hence in view of (15),

$$g(x)^2g'(y)^2-g'(x)^2g(y)^2=g'(0)^2[g(x)^2-g(y)^2]. \qquad (18)$$

If two equations are formed, respectively, by adding and by subtracting equations (16) and (17) and their product member by member is taken, the result readily reduces to

$$g(x+y).g(x-y).g'(x+y).g'(x-y)=g(x)^2g'(x)^2-g(y)^2g'(y)^2.$$

Divide the members of this equation by those of (15); then

$$g'(x+y).g'(x-y)=\frac{g(x)^2g'(x)^2-g(y)^2g'(y)^2}{g(x)^2-g(y)^2}$$

$$=g'(x)^2+g'(y)^2-\frac{g(x)^2g'(y)^2-g(y)^2g'(x)^2}{g(x)^2-g(y)^2}$$

$$=g'(x)^2+g'(y)^2-g'(0)^2,$$

in view of (18). Comparing this with (1) and (13) we have

$$g'(x)=\pm\frac{g'(0)}{2}(e^{kx}+e^{-kx}).$$

Therefore,

$$g(x)=\pm\frac{g'(0)}{2k}(e^{kx}-e^{-kx}, \text{ if } k=0, \tag{19}$$

since by (15) it is apparent that the constant of integration is zero. But

$$g(x)=\pm g'(0).x, \text{ if } k=0.$$

Therefore (15) has the two solutions (19) and (20), where $g'(0)$ is an arbitrary constant.

ALABAMA PRESBYTERIAN COLLEGE, *April, 1909.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

316. Proposed by B. F. FINKEL, Ph. D.

To prove that $\sum_{1}^{n}(-1)^{r-1}\frac{{}_rC_n}{r}=\sum_{1}^{n}\frac{1}{r}$.

II. Solution by S. LEFSEHETZ, East Pittsburg, Pa.

The proposition being true for $n=1$ and 2, the following is a proof by induction.

Suppose that $\sum_{1}^{n-1}(-1)^{r-1}\frac{1}{r}\cdot{}_rC_{n-1}=\sum_{1}^{n-1}\frac{1}{r}\ldots(1)$.

Since ${}_rC_{n-1}=\frac{n-r}{n}{}_rC_n$, therefore $\frac{1}{r}{}_rC_{r-1}=\frac{1}{r}{}_rC_n-\frac{1}{n}{}_1C_n$.

$$\therefore\sum_{1}^{n-1}(-1)^{r-1}\frac{1}{r}{}_rC_{n-1}=\sum_{1}^{n-1}(-1)^{r-1}\frac{1}{r}{}_rC_n-\frac{1}{n}\sum_{1}^{n-1}(-1)^{r-1}{}_rC_n=\sum_{1}^{n-1}\frac{1}{r}\ldots(2).$$

But $(1-1)^n=0=-\sum_{1}^{n-1}(-1)^{r-1}{}_rC_n-(-1)^{n-1}{}_nC_n+1.$

$$\therefore-\sum_{1}^{n-1}(-1)^{r-1}{}_rC_n=(-1)^{n-1}{}_nC_n-1.$$

By substituting in (2), we have, $\sum_{1}^{n}(-1)^{r-1}\frac{1}{r}{}_rC_n=\sum_{1}^{n}\frac{1}{r}$.

321. Proposed by C. C. BLAND, Attorney at Law, Rolla, Mo.

A corporation is capitalized for $20,000. 125 shares of the par value of $100 per share has been issued. A has 27 19/78 shares. B, C, D, E, and F each have 19 43/78 shares. It is the wish of the corporation to cancel the certificates held by A, B, C, D, E, and F, and to issue new certificates to each of them in lieu of those now held by them, and to avoid the issuance of any certificate for a fraction of a share. How many shares should each receive, the whole not to exceed 200, at the same time maintaining the present interest of each in the corporation?.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The value of $27\frac{19}{78}$ shares $=\$2724\frac{14}{39}=\$\frac{106250}{39}$, and the value of $19\frac{43}{78}$ shares $=\$1955\frac{5}{39}=\$\frac{76250}{39}$.

The greatest common divisor of these values is $\frac{1250}{39}$.

$\therefore \$\frac{1250}{39} = \$32\frac{2}{39}$ is the highest par value of each share in order that each may have whole shares and maintain his present interest.

A will then have 85 shares, B, C, D, E and E each 61 shares. Total number of shares in corporation, 624.

322. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Take six consecutive prime numbers, as 53, 59, 61, 67, 71, and 73, and find the least whole number such that if it be divided by 59 the remainder will be 53, if it be divided by 67 the remainder will be 61, and if it be divided by 73 the remainder will be 71, and show that this least whole number and the succeeding consecutive whole numbers that will fulfill this condition as to divisions and remainders are in arithmetical progression; and also show whether or not this is a general law for n consecutive prime numbers; and if there be such a general law whether or not that general law will lead to a general law for the finding of prime numbers.

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let x=the number. Then $\frac{x-53}{59}, \frac{x-61}{67}, \frac{x-71}{73}$=whole number.

Proceeding in the usual way, we get $x=27665-288569n$.

Therefore 27665 is the smallest whole number satisfying the condition. The succeeding numbers we get by putting $n=-1, -2, -3, \ldots$ which, of course, form an arithmetical progression the common difference of which is 289569. This will always be the case for any number of consecutive prime numbers, though it will not lead to a general law for the finding of prime numbers.

GEOMETRY.

347. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

ABC is a triangle, and D, E, F are the mid points of the arcs of its nine-point circle cut off by BC, CA, AB, respectively. The inscribed circle touches these sides at X, Y, Z. Are the lines DX, EY, FZ concurrent? A purely geometrical discussion required.

No solution of this problem has been received.

348. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Two parabolas and a rectangular hyperbola circumscribe a given quadrilateral. Find a relation between the squares of the latera recta of the parabolas and the squares of the perpendiculars from the center of the hyperbola to the axes of the parabolas.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

(1)... $m(x^2+y^2)+2nxy+2px+2qy+r=0$ is the equation of the rectangular hyperbola;

(2)... $(ax+by)^2+2gx+2fy+c=0$ is the equation of one parabola, and

(3)... $(Ax+By)^2+2Gx+2Fy+C=0$ is the equation of the other parabola

(4)... $[2(fa-gb)\sin^2\beta]/[a^2+b^2-2ab\cos\beta]^{\frac{1}{2}}$ is the latus rectum for (2);
(5)... $[2(FA-GB)\sin^2\rho]/[A^2+B^2-2AB\cos\rho]^{\frac{1}{2}}$ is the latus rectum for (3).

(6)... $(nq-mp)/(n^2-m^2)$, $(np-mq)/(n^2-m^2)$ is the center of (1).
(7)... $(a+b)(ax+by)+ag+bf=0$ is the axis of (2), and
(8)... $(A+B)(Ax+By)+AG+BF=0$ is the axis of (3).

β, ρ are the angles of inclination of the axes of (2), (3), respectively.

Let $PQRS$ be the quadrilateral; PQ, SR intersecting in the point T at an angle ω. Take T for the origin of all curves, and let $TP=h$, $TQ=h_1$, $TS=k$, $TR=k_1$. Then $\beta=\rho=\omega$.

(9) For (2), $h+h_1=-2g/a^2$, $hh_1=c/a^2$, $k+k_1=-2f/b^2$, $kk_1=c/b^2$.

(10) For (3), $h+h_1=-2G/A^2$, $hh_1=C/A^2$, $k+k_1=-2F/B^2$, $kk_1=C/B^2$.

(11) For (1), $h+h_1=-2p/m$, $hh_1=r/m$, $k+k_1=-2q/m$, $kk_1=r/m$.

$\therefore hh_1=kk_1$, $a=b$, $A=B$.

$g=-a^2(h+h_1)/2$, $f=-a^2(k+k_1)/2$, $G=-A^2(h+h_1)/2$, $F=-A^2(k+k_1)/2$. These values of g, f; G, F in (4), (5), respectively, give the same latus rectum for each. It is

$$\frac{(k+k_1-h-h_1)\sin^2\omega}{[2(1-\cos\omega)]^{\frac{1}{2}}}=\tfrac{1}{2}(k+k_1-h-h_1)\cot\tfrac{1}{2}\omega\operatorname{cosec}\tfrac{1}{2}\omega$$

$$=[(k-h)(h-k_1)\cot\tfrac{1}{2}\omega\operatorname{cosec}\tfrac{1}{2}\omega]/2h.$$

The square of the perpendicular from (6) on (7) is

$$(12)\ldots\ \frac{\left[\frac{p+q}{n+m}+\frac{g+f}{2a}\right]^2\sin^2\omega}{2(1-\cos\omega)}=\tfrac{1}{4}(h+h_1+k+k_1)^2\left(\frac{m}{n+m}+\frac{a}{2}\right)^2\cos^2\tfrac{1}{2}\omega.$$

The square of the perpendicular from (6) on (8) is

$$(13)\ldots\ \frac{\left[\frac{p+q}{n+m}+\frac{G+F}{2A}\right]^2\sin^2\omega}{2(1-\cos\omega)}=\tfrac{1}{4}(h+h_1+k+k_1)^2\left(\frac{m}{n+m}+\frac{A}{2}\right)^2\cos^2\tfrac{1}{2}\omega.$$

(12) $\div$ (13) gives $\left[\dfrac{m(2+a)+an}{m(2+A)+An}\right]^2$ for the ratio.

CALCULUS.

279. Proposed by L. H. McDONALD, M. A., Ph. D., Sometime Tutor at Cambridge, Jersey City, N. J.

Find the ellipse of minimum area which will pass through the vertices of a triangle. (Hedrick-Goursat's *Math. Anal.*, p. 133, ex. 9.)

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

If B be the origin; BA, BC the axes; then from Vol. X, Nos. 8-9, of the MONTHLY, page 204,

$$clx^2+any^2+(al+cn-bm)xy-aclx-acny=0,$$

is the equation to the circum-ellipse, and

$$\frac{2\pi a^2bc^2lmn\sin B}{[4acln-(al+cn-bm)^2]^{\frac{3}{2}}}=\text{its area.}$$

Let $m=ul$, $n=vl$. $\quad\therefore cx^2+avy^2+(a+cv-bu)xy-acx-acvy=0\ldots(1).$

$$\frac{2\pi a^2bc^2uv\sin B}{[4acv-(a+cv-bu)^2]^{\frac{3}{2}}}=m=\text{minimum.}$$

Now $dm/du=0$ and $dm/dv=0$.

$$\therefore 4acv-(a+cv-bu)^2=3bvu(a+cv-bu),\ \ldots(2).$$
$$4acv-(a+cv-bu)^2=3cvu(a-cv+bu).\ \ldots(3).$$

Hence, $b(b+c)u=c(b+c)v+a(b-c)$, or $u=\dfrac{cv}{b}+\dfrac{a(b-c)}{b(b+c)}\ldots(4).$

(4) in (2) gives, $3cv^2(b+c)-v[2(b+c)^2-3a(b-c)]+2ac=0.$

$$\therefore v=\frac{2(b+c)^2-3a(b-c)-\sqrt{[2(b+c)^2-3a(b-c)]^2-24ac^2(b+c)}}{6c(b+c)}.$$

$$u=\frac{2(b+c)^2+3a(b-c)-\sqrt{[2(b+c)^2-3a(b-c)]^2-24ac^2(b+c)}}{6c(b+c)}.$$

These values of u and v in (1) give the required ellipse.

Corollary. If $a=b=c$, $u=v=\frac{1}{3}$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let ACB be the triangle; choose $BC=a$ for the axis of abscisses, and CA for that of ordinates. Any circumscribed ellipse is of the form $y^2+Axy+Bx^2+Cy+Dx=0$; and since $(a, 0)$ and $(0, b)$ are points of the ellipse, we have $C=-b$, $D=-Ba$, and the above equation reduces to $y^2+Axy+Bx^2-by-Bax=0$.

Transforming it to the center of the ellipse as origin, it reduces to

$$y^2+Axy+Bx^2-\frac{Bb^2-AB\,ab+B^2a}{4B-A^2}=0.$$

The area is $=\pi \sin C.\ \frac{Bb^2-AB\,ab+B^2a}{4B-A^2}=m$. Developing $\frac{\partial m}{\partial A}=0$, and $\frac{\partial m}{\partial B}=0$, we find $A=\frac{b}{a}$ and $B=\frac{b^2}{a^2}$, and thus find the ellipse of minimum area to be, $a^2y^2+abxy+b^2x^2-a^2by-ab^2x=0$. The center is the point $(a/3, b/3)$. The maximum ellipse about the triangle is concentric, and its equation is

$$a^2y^2+abxy+b^2x^2-a^2by-ab^2x+\frac{a^2b^2}{4}=0.$$

Also solved by C. N. Schmall, and V. M. Spunar.

280. Proposed by C. N. SCHMALL, 89 Columbia Street, New York.

Find the envelope of the system of spheres

$$\left.\begin{aligned}(x-a)^2+(y-b)^2+z^2&=r^2\\ a^2+b^2&=c^2\end{aligned}\right\}.$$

Solution by J. SCHEFFER, A. M., Hagerstown, Md.; G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; V. M. SPUNAR, Pittsburg, Pa., and the PROPOSER.

Differentiating $(x-a)^2+(y-b)^2+z^2=r^2$ and $a^2+b^2=c^2$ with reference to a as the independent variable, we have $(x-a)+(y-b)\frac{\partial b}{\partial a}=0$, and $a+b\frac{\partial b}{\partial a}=0$; from the second equation we get $\frac{\partial b}{\partial a}=-\frac{a}{b}$, and substituting in the first we get $(x-a)-\frac{a}{b}(y-b)=0$, whence $b=\frac{ay}{x}$ and combining this with $a^2+b^2=c^2$, we get $a^2=\frac{c^2x^2}{x^2+y^2}$, $b^2=\frac{c^2y^2}{x^2+y^2}$, and substituting this in the first given equation, we have, after some easy reductions:

$$x^2+y^2+z^2-2c\sqrt{(x^2+y^2)}=r^2-c^2.$$

This is, as can easily be seen, the equation of the surface of a ring, the central line of which is a circumference whose radius is $=c$, and the perpendicular section of a circle whose radius is $=\sqrt{(2c^2-r^2)}$.

MECHANICS.

229. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Find the position of the center of pressure of a semi-elliptical area completely immersed in water, the area being vertical, the bounding axis major being inclined to the horizon at an angle β, and having one extremity in the surface of the water.

Solution by the PROPOSER.

Let AB be the intersection of the vertical plane with the surface of the water; AC the major axis of the ellipse$=2a$; Q any point on the area of the semi-ellipse; $QD=h$, the perpendicular distance from Q to AB; $\angle BAC=\beta$, $\angle CAQ=\theta$, $AQ=r$. Then

$$r=\frac{2ab^2\cos\theta}{a^2\sin^2\theta+b^2\cos^2\theta},\quad h=r\sin(\theta+\beta).$$

$$\bar{x}=\frac{\int_0^{\frac{1}{2}\pi}\int_0^r hr^2\cos\theta\,d\theta\,dr}{\int_0^{\frac{1}{2}\pi}\int_0^r hr\,d\theta\,dr},\quad \bar{y}=\frac{\int_0^{\frac{1}{2}\pi}\int_0^r hr^2\sin\theta\,d\theta\,dr}{\int_0^{\frac{1}{2}\pi}\int_0^r hr\,d\theta\,dr}.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^r hr\,d\theta\,dr=\int_0^{\frac{1}{2}\pi}\int_0^r r^2\sin(\theta+\beta)d\theta\,dr=\tfrac{1}{3}\int_0^{\frac{1}{2}\pi}r^3\sin(\theta+\beta)d\theta=N.$$

$$\therefore N=\frac{8a^3b^6}{3}\int_0^{\frac{1}{2}\pi}\frac{(\cos^3\theta\,\tin\theta\cos\beta+\cos^4\theta\sin\beta)d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^3}$$

$$=\tfrac{1}{6}ab(4b\cos\beta+3a\pi\sin\beta).$$

$$M=\int_0^{\frac{1}{2}\pi}\int_0^r hr^2\cos\theta\,d\theta\,dr=\int_0^{\frac{1}{2}\pi}\int_0^r r^3\cos\theta\sin(\theta+\beta)d\theta\,dr$$

$$=\tfrac{1}{4}\int_0^{\frac{1}{2}\pi}r^4\cos\theta\sin(\theta+\beta)d\theta$$

$$=4a^4b^8\int_0^{\frac{1}{2}\pi}\frac{(\cos^5\theta\sin\theta\cos\beta+\cos^6\theta\sin\beta)d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^4}$$

$$=\tfrac{1}{24}a^2b(16b\cos\beta+15a\pi\sin\beta).$$

$$P=\int_0^{\frac{1}{2}\pi}\int_0^r hr^2\sin\theta\,d\theta\,dr=\int_0^{\frac{1}{2}\pi}\int_0^r r^3\sin\theta\sin(\theta+\beta)d\theta\,dr$$

$$= \tfrac{1}{4}\int_0^{\frac{1}{2}\pi} r^4 \sin\theta \sin(\theta+\beta)\, d\theta$$

$$= 4a^4b^8 \int_0^{\frac{1}{2}\pi} \frac{(\cos^4\theta \sin^2\theta \cos\beta + \cos^5\theta \sin\theta \sin\beta)\, d\theta}{(a^2\sin^2\theta + b^2\cos^2\theta)^4},$$

$$= \tfrac{1}{2^{\frac{1}{2}}}ab^2 (16a\sin\beta = 3\pi b\cos\beta).$$

$$\bar{x} = M/N = \tfrac{1}{4}a.\frac{16b\cos\beta + 15a\pi\sin\beta}{4b\cos\beta + 3a\pi\sin\beta}, \quad \bar{y} = P/N = \tfrac{1}{4}b.\frac{16a\sin\beta + 3b\pi\cos\beta}{4b\cos\beta + 3a\pi\sin\beta}.$$

If $\beta = 0$, $\bar{x} = a$, $\bar{y} = 3\pi b/16$.

Also solved by S. Lefschetz.

230. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A particle is projected from a distance $a = 2r$ from the earth's center towards the earth with a velocity from infinity. If the earth were an airless homogeneous sphere, radius equal to the present mean radius and gravity as at present, with what velocity and in what time would it reach the center through an opening from surface to center?

Solution by the PROPOSER.

$\int_{v_1}^{v} v\,dv = \int_x^a f\,dx$, but $f = \frac{gr^2}{x^2}$ above the earth where r is earth's radius.

$$\therefore \int_{v_1}^{v} v\,dv = gr^2 = \int_x^a \frac{dx}{x^2}. \quad \therefore \tfrac{1}{2}(v^2 - v_1^2) = \frac{gr^2(a-x)}{ax}.$$

$$\therefore v^2 = \frac{2gr^2(a-x)}{ax} + v_1^2 = \left(\frac{dx}{dt}\right)^2, \text{ where } v_1 \text{ is the initial velocity.}$$

$$\therefore t = \sqrt{a}\int_r^a \frac{\sqrt{(x)}\,dx}{\sqrt{[2agr^2 + x(av_1^2 - 2gr^2)}}.$$

If $v_1 = r\sqrt{\frac{2g}{a}}$, $t = \frac{2(a^{\frac{3}{2}} - r^{\frac{3}{2}})}{3r\sqrt{(2g)}}$, $v = \sqrt{(2gr)}$ at surface.

If $v_1 < r\sqrt{\frac{2g}{a}}$, $t = 2gr^2\sqrt{\left(\frac{a}{2gr^2 - av_1^2}\right)^3}\left(\cos^{-1}\sqrt{\frac{2gr^2 - av_1^2}{2agr}}\right.$

$$\left. - \cos^{-1}\sqrt{\frac{2gr^2 - av_1^2}{2gr^2}} + \frac{\sqrt{[(2gr^2 - av_1^2)(2agr + av_1^2 - 2gr^2)]}}{2agr}\right.$$

$$-\frac{v_1\sqrt{[a\,(2gr^2-av_1^2)]}}{2gr^2}\Big).$$

$$\text{If } v_1<r\sqrt{\frac{2g}{a}},\ t=2gr^2\sqrt{\left(\frac{a}{av_1^2-2gr^2}\right)^3}\left\{\frac{v_1\sqrt{[a(av_1^2-2gr^2)]}}{2gr^2}\right.$$

$$-\frac{\sqrt{[(av_1^2-2gr^2)\,(2agr+av_1^2-2gr^2)]}}{2agr}$$

$$\left.+\log\left[\sqrt{\frac{a}{r}}\left(\frac{\sqrt{(av_1^2-2gr^2)}+v_1\sqrt{a}}{\sqrt{(av_1^2-2gr^2)}+\sqrt{(2agr+av_1^2-2gr^2)}}\right)\right]\right\}.$$

Since $v_1=\sqrt{(2gr)}$ = velocity from infinity, and $a=2r$, $v_1>r\sqrt{(2g/a)}$. Hence time of falling to surface is

$$t=2\sqrt{\frac{r}{g}}\left(\frac{2\sqrt{2}-\sqrt{3}}{2}+\log\frac{2+\sqrt{2}}{1+\sqrt{3}}\right).$$

Velocity at the surface becomes

$$v_2^2=2gr^2\left(\frac{2r-r}{2r^2}\right)+2gr=3gr. \quad \therefore v_2=\sqrt{(3gr)}.$$

The formula for below the earth's surface is

$$\int_{v_2}^{v_3}v_3dv=\int fdx=\frac{g}{r}\int_0^x(r-x)\,dx. \quad \therefore v_3^2-v_2^2=\frac{g}{r}(2rx-x^2).$$

$$\therefore v_3^2=(g/r)\,(2rx-x^2)+v_2^2=(dx/dt)^2.$$

$$\therefore t_1=\int_0^r\frac{dx}{\sqrt{[v_2^2+(g/r)\,(2rx-x^2)]}}=\sqrt{\frac{r}{g}}\sin^{-1}\left(\frac{r\sqrt{g}}{\sqrt{(v_2^2r+gr^2)}}\right).$$

Hence time of falling from surface to center is $t_1=\frac{\pi}{6}\sqrt{\frac{r}{g}}$.

Since $v_2=\sqrt{(3gr)}$, $v_3=2\sqrt{(gr)}$, the velocity at the center when $x=r$.

Taking for the mean semi-axes of the earth the values 20926202 feet and 20854895 feet, the radius of a sphere of equal volume is 20902410 feet, and the acceleration g at the surface of this sphere is 32.10614 feet.

$\therefore \sqrt{(r/g)}=806.871.$

$$T=t+t_1=\sqrt{\frac{r}{g}}\left(\tfrac{1}{6}\pi+2\sqrt{2}-\sqrt{3}+2\log\frac{2+\sqrt{2}}{1+\sqrt{3}}\right)=1666.673 \text{ seconds} = 27$$

minutes, 46.673 seconds.

v_3=44869.668 feet per second=8.498 miles=8½ miles per second, nearly.

When $v_1=r\sqrt{(2g/a)}$, $v_2=\sqrt{(2gr)}$, and the velocity of arriving at the surface is independent of a, the distance from the center.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

163. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Prove that the equation $y^n=mx+1$ always has at least one positive integer solution (different from $y=1$, $x=0$), whatever integer values m and n may have.

Solution by S. LEFSEHETZ, Pittsburg, Pa.

The following solution is evident:

$$y=(m+1)^n,\quad x=\frac{(m+1)^n-1}{m}.$$

To find all solutions, we remark that it is enough to find all values of y such that $y^n\equiv 1 \pmod m$. Let f be a divisor of $\phi(m)$, a a number to which f appertains. Then $a^f\equiv 1 \pmod m$. If also $a^n\equiv 1 \pmod m$, we must have $n\equiv 0 \pmod f$. Hence, f is a divisor of $dv[n, \phi(m)]$. Therefore we take the $\phi(m)$ numbers smaller than m and prime to it, we form the exponents to which they appertain and keep them if their exponents divide $dv[n, \phi(m)]$. If x be such a value, $y=x+km$ is a solution, the corresponding value of x being $\frac{y^n-1}{m}$, which is integral since $y^n\equiv 1 \pmod m$.

AVERAGE AND PROBABILITY.

200. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

A line $AB=l$ is extended to P making $BP=p$. If a point D is taken at random in BP, what is the mean value of $AD.DP$?

Solution by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

Let $x=BD$. Then $AD=l+x$, $DP=p-x$, and $AD.DP=(l+x)(p-x)$.

$$\therefore M=\int_0^p (l+x)(p-x)\,dx \Big/ \int_0^p dx = \frac{1}{p}\int_0^p [lp+(p-l)x-x^2]\,dx$$

$$=\tfrac{1}{6}p(3l+p).$$

Also solved by G. B. M. Zerr.

201. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A random straight line is drawn across a circle and another through a given point on the circumference. Find the chance that they intersect within the circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let θ be the angle the line through a given point on the circumference makes with the diameter through the point, AB=length of this chord.

Then for favorable case of intersection the random line must intersect $AB=2a\cos\theta$ where a=radius of circle.

$$\therefore \text{Chance}=\int_0^{\frac{1}{2}\pi} 2a\cos\theta\, d\theta \Big/ \int_0^{\pi} 2a\, d\theta = \int_0^{\frac{1}{2}\pi} \cos\theta\, d\theta \Big/ \int_0^{\pi} d\theta = \frac{1}{\pi}.$$

Also solved by the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

326. Proposed by R. D. CARMICHAEL, Princeton University.

Is the series, of which the nth term is $\dfrac{1.3.5.7\ldots(2n-1)}{(n+1)!\,2^n.(2n+3)}$ convergent? If so, find its sum.

327. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

The coefficients of the algebraical equation $f(x)=0$ are all integers. Show that if $f(0)$ and $f(1)$ are both odd numbers, the equation can have no integral roots.

328. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

If $x^2+xy+y^2=3a^2$, find the maximum value of $bx+cy$.

GEOMETRY.

354. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Find the condition that triangles which are circumscribed to one of two confocal parabolas may be inscribed in the other.

355. Proposed by JOHN J. QUINN, New Castle, Pa.

If an indefinite line MC cuts the Y-axis at A a fixed point, and the X-axis at B, and at its extremity C another line $PCDP'$ be pivoted cutting the X-axis at D and extending to P', so that $PC=CD=BC$, and $PD=P'D$: (1) Find the locus of P and P' as MC slides through A; (2) Apply to the trisection of an angle; (3) Prove PP' a constant tangent to upper branch; (4) Show condition which gives rise to loop; (5) Show its relation to conchoid; (6) Discuss for other properties.

356. Proposed by G. I. HOPKINS, Manchester, N. H.

Required to construct a triangle having given, base, vertical angle, and difference of other two sides.

CALCULUS.

284. Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.

Inscribe the triangle of maximum area in a given circle.

285. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

If R_1 and R_2 are the radii of curvature of an ellipse at the extremities of a pair of conjugate diameters, show that $R_1^{2/3}+R_2^{2/3}=\frac{a^2+b^2}{(ab)^{2/3}}$, where a, b, are the semi-axes.

286. Proposed by R. D. CARMICHAEL, Princeton University.

Solve the differential equation

$$\begin{aligned}&[a_0x^3+a_1x^2y+a_2xy^2+(a_0-a_1+a_2)y^3\\&\qquad+a_3x^2+a_4xy+a_5y^2+a_6x+a_7y+a_8]dx\\&+[a_0y^3+a_1xy^2+a_2x^2y+(a_0-a_1+a_2)x^3\\&\qquad+a_3y^2+a_4xy+a_5x^2+a_6y+a_7x+a_8]dy=0.\end{aligned}$$

MECHANICS.

238. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Find the position of the center of pressure of a semi-elliptical area completely immersed in water, the bounding major-axis being inclined to the horizon at an angle β, and having one extremity in the surface of the water.

239. Proposed by J. G. ROSE, B. A. (Oxion), Mt. Angel College, Oregon.

A uniform bar of length $2a$ is placed in a sloping position, its lower end on the ground (coefficient of friction being μ), its upper end in the air, the bar being supported by a rough fixed peg (coefficient of friction μ'), against which it rests. If h is the height of the peg from the ground, and if θ be the angle the bar makes with the horizon, when on the point of slipping, prove that θ is to be found from the equation

$$\sin\theta\cos\theta\,[(\mu-\mu')\cos\theta+\sin\theta\,(1+\mu\mu')]=\mu\,h/a.$$

240. Proposed by S. A. COREY, Hiteman, Iowa.

A perfectly flexible wire rope weighing one pound per foot is suspended from the tops of two vertical supports 300 feet apart, one support being 30 feet higher than the other. One end of the rope is fastened to the top of the higher support, while 600 feet of the rope hangs vertically from the top of the lower support. Assuming that the rope is free to slide over the top of the lower support without friction, find the lowest point of that portion of the rope which is suspended between the supports. Also find the amount of work which must be performed in raising the lowest point to make it coincide with the top of the lower support by exerting a pull on the free end of the rope.

NOTES AND NEWS.

At the ninth annual meeting of the Central Association of Science and Mathematics Teachers, a committee appointed the year preceding made an extended report on the securing of real, applied problems in algebra and geometry suitable for use in secondary schools. A large number of teachers had contributed problems during the year and these had been printed in *School Science and Mathematics.* It was pointed out, however, that in many such problems the technical information necessary made them unsuitable for young students, and even problems which might be understood by boys would, in most cases, be little adapted to girls. For this reason most technical problems drawn from carpentry, masonry, machine shops, etc., are of doubtful practical application in the schoolroom. But it was also pointed out that problems in geometry drawn from architecture, decorative and ornamental design are less technical in character and are equally well adapted to both boys and girls.

This question, of vital importance in secondary work, is also commanding attention in college courses and is materially affecting the character of college texts, especially in the calculus.

Another committee reported on the question of unified mathematics for secondary schools, a subject which is also of interest in respect to college mathematics. S.

BOOKS.

Descriptive Geometry. A Treatise from a Mathematical Standpoint, together with a Collection of Exercises and Practical Applications. By Victor T. Wilson, M. E., Professor of Drawing and Design, Michigan Agricultural College. 8vo, viii+237 pages, 140 figures. Cloth, $1.50 net. New York: John Wiley & Sons.

Descriptive geometry is essentially a mathematical subject. The application of its principles to the making of working drawings, however, and the modifications which are made to suit the contingencies of practice, have had a tendency to obscure this fact, and like other theoretical subjects it has suffered mutilation in the interest of short cuts to immediate practical uses. But does not technical education, after all, consist chiefly in an equipment of sound theory? It has been the author's purpose to refrain from any attempt to hold the student's interest by clothing a few principles with some immediate practical application, but instead, to present a sound theoretical treatment. How well he has succeeded he leaves others to judge.

The principles are herein formulated under theorems, as to plane and solid geometry; Illustrative problems are solved in accordance with these theorems and special constructions discussed. The plan of, at least, one well known text is followed of dividing all problems in two parts; the first of which is a statement of the geometrical principles and

the theoretical solution called an analysis; the second is a description of a graphic solution, accompanied by a drawing. An important feature is added, however, of giving the statement of the geometrical conditions and the solution in the analysis in a general form, instead of being made to refer to a certain kind of problem exclusively. *From the Preface.*

The Slide Rule. An Elementary Treatise. By J. J. Clark, M. E. (Lehigh), Dean of the Faculty, International Correspondence Schools, Manager of the Text-book Department, International Text-book Co. Cloth, 62 pages. Scranton, Pa.: Technical Supply Co.

In this little volume, are set forth in detail the use and applications of the slide rule. As many practical men are using the slide rule, this little volume will enable them to extend their knowledge of its application. F.

College Algebra. By H. L. Rietz, Ph. D. (Cornell), Assistant Professor of Mathematics, University of Illinois, and A. R. Carthorne, Ph. D. (Goettingen), Associate Professor of Mathematics, University of Illinois. 8vo. Cloth, xiii+261 pages. Price, $1.40. New York: Henry Holt & Co.

The book begins with a review of High School Algebra, for the benefit of the student who, having had his high school algebra two years before entering college, requires a hasty review of first principles. However, it is not all review; some matter, for example, determinants and graphs, is introduced in order to give the student, at the outset, an enlarged conception of the subject. Many problems from physics and engineering are introduced, yet in this particular the work is not overdone. The book is one of merit, and will lend itself readily to successful teaching. F.

A Text-book of General Physics for Colleges: Mechanics and Heat. By J. A. Culler, Ph. D., Professor of Physics, Miama University. 8vo. Cloth, ix+311 pages. Price, $1.80. Philadelphia: J. B. Lippincott & Co.

In this book, the principles of Mechanics and Heat are set forth in clear and simple terms. The type is large and the illustrations good. Reference matter and tables are placed in an appendix. A number of short lists of problems are found where needed to illustrate principles, and the answers are given at the end of the lists. A table of sines, cosines, tangents, etc., are also appended. The book will be serviceable to those teachers who prefer texts dealing with specific topics. F.

Dynamical Theory of the Capture of Satellites and of the Division of Nebulae under the Secular Action of the Resisting Medium. By T. J. J. See. Reprinted from the *Astr. Nach.*, Vol. 181.

The author has attempted to show by mathematical reasoning that the solar system had its origin, not by the formation of the planets and satellites detached from a central mass as was assumed by LaPlace, but that these planets and satellites were captured from without and have since had their orbits reduced in size and rounded up under the secular action of a resisting medium. The article is interesting even though one might not agree with all the assumptions necessary to establish the conclusions to which the author's reasoning leads. F.

ERRATA.

Page 174, line 24, for "have been printed" read "had been printed."
Page 176, line 30, for "like questions" read "like question."
Page 177, line 16, for "improvements" read "improvement."

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NOTE ON A PROOF IN CHRYSTAL'S ALGEBRA.

By T. E. McKINNEY, University of South Dakota.

An analytic proof of the "fundamental theorem of algebra" is given in Chrystal's *Text-book of Algebra*, Part I, 5th Ed., Chap. XII. In view of the character and authority of this admirable treatise it may be worth while to call in question one or two points in this proof.

The argument, in outline, is substantially this: It is first established, in effect, that the rational, integral function $f(w)$ of degree n in w vanishes, if at all, within a circle, S, having a radius R and the origin for center. It is then virtually assumed that within this circle S, $|f(w)|$ has a lower limit L. The further assumption is made that $L>0$, and the argument is arranged to show that this latter assumption involves a contradiction. To do this, a point w is taken within S, such that

$$|f(w)| = L+\varepsilon,\ \varepsilon>0.$$

A neighboring point, $w+h$, is then taken for the purpose of showing that

$$|f(w+h)| < L,$$

when the h is properly chosen. Now, h having been suitably determined, the argument proceeds:

"$\left|\frac{f(w+h)}{f(w)}\right|$ will lie between two positive proper fractions, so that

$$\left|\frac{f(w+h)}{f(w)}\right| = 1-\mu,$$

where μ is a proper fraction; and we have

$$|f(w+h)| = (1-\mu)\,|f(w)| = (1-\mu)(L+\varepsilon) = L+\varepsilon-\mu(L+\varepsilon) \qquad (7)$$